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# On normal ordering and canonical transformations in thermal field theory

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**Abstract.** We look at a real scalar field in thermal equilibrium in the context of the new normal ordering and field split defined by Evans and Steer (Evans T S and Steer D A 1996 *Nucl. Phys.* B **474** 481). We show that the field split defines a natural canonical transformation, but that this transformation differs from others known in thermal field theory.

## 1. Introduction

In a recent work [1], the concept of normal ordering in path-ordered approaches to thermal field theory [2–6] was discussed. In particular it was shown that with a new definition, one can ensure that the thermal expectation value of all normal-ordered products are zero. This is true for all types of fields and for all contours in the complex time plane, including Matsubara's imaginary time contour [2]. As a result, the canonical derivation of quantum field theory (QFT) at finite temperature,  $T$ , can be seen to proceed just as at zero temperature. In [1] normal ordering was defined in terms of the split of the field, say  $\psi(x)$ , into 'positive' and 'negative' parts:

$$\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x). \quad (1)$$

In the following discussion we keep the labels 'positive' and 'negative' even though, as was shown in [1], the preferred  $T > 0$  split is more general than the usual  $T = 0$  split and is *not* a split into positive and negative energy waves.

We remark that the use of a more general split is natural since in QFT the main task of field splitting is not so much the separation of the field into positive and negative energy waves, but its separation into annihilation and creation operators. Thus the real problem is which *definition* of the annihilation and creation operators to use in order to best describe the physical system under consideration. In other words, it is important to choose the correct physical representation of the underlying algebraic structure of the theory, since the representations have a different physical content and the algebra alone does not determine the annihilation and

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creation operators. The fundamental property of QFT is that there are infinitely many such representations, and this allows it to represent a wide variety of physical phenomena [7].

There are many examples of this. In spontaneous symmetry breaking, the value of the order parameter *selects* the appropriate physical *vacuum* and thus the appropriate representation of the canonical commutation relations. This means that the irreducible set of physical fields is not given *a priori*, but is dictated by the symmetry breaking condition [7]. In a similar way, in flavour oscillations (such as in kaon or neutrino systems) a non-trivial choice of appropriate physical vacuum is made [8]. This problem must also be faced when working with quantum fields in curved space time [9, 10] or when studying the canonical quantization of gravitational wave mode evolution in inflating universe [11]. Closely related is the need to pick representations carefully when quantizing dissipative systems [12]. In all the above cases, Bogoliubov transformations [13] play a central role, and the new creation and annihilation operators defined by such transformations are invariably related by hermitian conjugation<sup>†</sup>. Finally, we note that careful attention is needed to define the *proper* creation and annihilation operators in the quantization of two-dimensional gravity models [14].

Returning to the thermal field theory context, it is well known that the very same particle concept loses any meaning at non-zero temperature [15, 16]. There is, therefore, an intrinsic physical relevance in the study of how to define and to deal with *particle* creation and annihilation operators, if any, at finite temperature, and in the understanding of to what extent it is possible to associate a meaning to them in terms of physical excitations. In a word, the problem of how to construct a formalism which is ‘canonical’ is a crucial, and in some respects an urgent problem to solve. The discussion we present in this paper is a contribution in such a direction. Our standpoint is that, in a similar way to classical mechanics and to zero temperature QFT, searching for and studying the properties of the canonical transformations of the theory is a priority in order to keep contact with the physical content of the formalism.

It is well known that in thermal field theory, Bogoliubov transformations are central to the approach known as thermo field dynamics (TFD) [3–5, 16, 17]. There the correct choice of vacuum depends on the temperature, but otherwise the derivation is very similar to the canonical approach to zero temperature QFT. An interesting point to note—which will become relevant later—is that in TFD it is possible to work with a pair of canonical operators which are *not* hermitian conjugates ( $\alpha \neq \frac{1}{2}$  in the notation of [16]). Such non-hermitian representations give the same physical results at equilibrium and indeed one ( $\alpha = 1$ ) is favoured both in and out of equilibrium [5, 16].

Given the prevalence of Bogoliubov transformations in quantum field theory and their central role in the TFD approach to thermal field theory, it is surprising that no such structure has been found in the alternative path-ordered approaches to thermal field theory, which is the context of the new normal-ordered product of [1]. The path-ordered formalisms are all based on a contour in the complex time-plane which ends  $-i\beta$  below its starting point [2–6]. They come in two varieties, real- and imaginary-time formalisms. The real-time versions are distinct from TFD [4] (for example the various fields and their creation and annihilation operators have distinct properties) but perturbatively they are completely equivalent in equilibrium, thus reflecting an underlying relationship [5, 18]<sup>‡</sup>. In the real-time path-ordered approaches a thermal Bogoliubov transformation appears, but only post hoc, say when the structure of the propagator is analysed. However, there is no sign of a thermal Bogoliubov transformation in the imaginary-time approaches.

<sup>†</sup> In some of the above situations, since the Lorentz (Poincaré) covariance is lost, the notion of vacuum (and thus of field splitting into positive and negative frequency parts) is in fact missing. And this brings us back to the problem of the *definition* of the annihilation and creation operators [9] and of the *proper* normal ordering.

<sup>‡</sup> This similarity has led to confusion in the literature over nomenclature.

The split discovered in [1] by considering normal ordering in path-ordered TFT combines annihilation and creation operators in a way that is reminiscent of Bogoliubov transformations used in other problems. The aim of this paper is to elucidate the precise relation between the split of [1] and canonical transformations. In section 2, we introduce our notation and summarize the results of [1] focusing on real scalar fields. In section 3 we search for a canonical structure in those results. This structure is then put in the context of Bogoliubov transformations in section 4. In section 5 we present our conclusions, and discuss extensions of our work to other fields.

## 2. Normal ordering and field splitting

We consider real scalar relativistic fields

$$\phi(x) = \sum_k \frac{1}{\sqrt{2\omega_k V}} (a_k(t)e^{ik \cdot x} + a_k^\dagger(t)e^{-ik \cdot x}) \tag{2}$$

where  $a_k(t) = e^{-i\omega_k t} a_k$  and  $V$  is the volume of the system. The precise form of the dispersion relation is unimportant; we only require that  $\omega_k = \omega_{-k}$ . The annihilation and creation operators  $a_k, a_k^\dagger$  obey the canonical commutation relations

$$[a_k, a_{k'}^\dagger] = \delta_{k,k'} \tag{3}$$

and all other commutators are zero. The vacuum state for  $a_k$  is denoted by  $|0\rangle$ ;  $a_k|0\rangle = 0$ . We consider thermal equilibrium and denote thermal averages by double angular brackets:

$$\langle\langle \dots \rangle\rangle = \frac{1}{Z} \text{Tr}\{e^{-\beta H} \dots\}. \tag{4}$$

Here  $\dots$  means any operator,  $H$  is the Hamiltonian,  $Z$  is the partition function and  $\beta$  is the inverse temperature:  $\beta = 1/T$  ( $k_B = 1$ ). The trace is over a complete set of states for the system. Since we have no chemical potential

$$\langle\langle a_k^\dagger a_p \rangle\rangle = n_k \delta_{k,p} \quad \langle\langle a_k a_p^\dagger \rangle\rangle = (1 + n_k) \delta_{k,p} \tag{5}$$

where  $n_k$  is given by the Bose–Einstein distribution

$$n_k = \frac{1}{e^{\beta\omega_k} - 1}. \tag{6}$$

In [1] normal ordering was defined in terms of the arbitrary split (1), so generalizing the traditional  $T = 0$  definition. We *always* define normal ordering to strictly mean that all (+)-fields are moved to the right of (–)-fields; otherwise the order of the fields is left unchanged. For example,

$$N[\phi_1 \phi_2] = \phi_1^{(+)} \phi_2^{(+)} + \phi_1^{(-)} \phi_2^{(+)} + \phi_2^{(-)} \phi_1^{(+)} + \phi_1^{(-)} \phi_2^{(-)} \tag{7}$$

where  $\phi_i = \phi(x_i, t_i)$ .

Using this generalized definition of normal ordering, it was shown in [1] that Wick’s theorem holds in its usual form *if* a split is chosen such that the contraction is a  $c$ -number. This is satisfied by splits (1) which are linear in the annihilation and creation operators and only these were considered in [1]. It was then shown that if the fields are split such that the thermal expectation value of two-point normal-ordered products vanish, then the thermal expectation value of all  $n$ -point normal-ordered products vanish.

The only splits which guarantee that  $\langle\langle N[\phi_1 \phi_2] \rangle\rangle = 0$  for all times were shown in [1] to be

$$\phi^{(+)}(x) = \sum_k \frac{1}{\sqrt{2\omega_k V}} [(1 - f_k) a_k e^{-ik \cdot x} + g_k a_k^\dagger e^{ik \cdot x}] \tag{8}$$

$$\phi^{(-)}(x) = \sum_k \frac{1}{\sqrt{2\omega_k V}} [f_k a_k e^{-ik \cdot x} + (1 - g_k) a_k^\dagger e^{ik \cdot x}] \tag{9}$$

where there are two solutions for  $f_k$  and  $g_k$ , namely

$$f_k = -n_k + s_k[n_k(n_k + 1)]^{1/2} \quad g_k = -n_k - s_k[n_k(n_k + 1)]^{1/2} \quad (10)$$

$$s_k = \pm 1. \quad (11)$$

Note that  $s_k$  can be chosen to be a function of both the size *and* direction of  $k$ .

These solutions were derived within the context of the path-ordered approaches to thermal field theory. The solution is independent of the curve chosen and so all the work in [1] and in the present paper applies to both imaginary-time and all real-time path-ordered approaches to thermal field theory.

### 3. Searching for canonical transformations

The split (8), (9) contains factors which are reminiscent of those seen in Bogoliubov transformations, and particularly those encountered in TFD. It is therefore interesting to see if this new split defines any new canonical operators and if so, whether those operators can be given any physical significance.

We may rewrite the split (8), (9) as

$$\phi^{(+)} = \sum_k \frac{d_k}{\sqrt{2\omega_k V}} e^{+ik \cdot x} X_k(t) \quad \phi^{(-)} = \sum_k \frac{d_k}{\sqrt{2\omega_k V}} e^{-ik \cdot x} X_k^{\flat}(t) \quad (12)$$

where  $d_k$  is a normalization factor to be determined later, and the new operators  $X_k$  and  $X_k^{\flat}$  are given by

$$X_k(t) = e^{-i\omega_k t} \frac{(1 - f_k)}{d_k} a_k + e^{+i\omega_k t} \frac{g_k}{d_k} a_{-k}^{\dagger} \quad (13)$$

$$X_k^{\flat}(t) = e^{-i\omega_k t} \frac{f_k}{d_k} a_{-k} + e^{+i\omega_k t} \frac{(1 - g_k)}{d_k} a_k^{\dagger}. \quad (14)$$

Note that the new definition of normal ordering, (7)–(9), is now equivalent to the rule of putting all ‘annihilation’ (‘creation’) operators  $X_k$  ( $X_k^{\flat}$ ) to the right (left), just as we are used to at zero temperature. We know from the work of [1] that this will guarantee that the thermal expectation value of any normal-ordered product of fields vanishes. Thus for this finite temperature system,  $X_k$  and  $X_k^{\flat}$  seem to mimic the usual creation and annihilation operators.

We have introduced a new operation, which we call ‘flat conjugation’ and denote with a  $\flat$  symbol<sup>†</sup>. Flat conjugation is defined by (14) and it consists of both hermitian conjugation and the exchange  $f \leftrightarrow g$ :

$$A^{\flat}[f, g] = (A[f, g])^{\flat} \equiv (A[g, f])^{\dagger} \quad (15)$$

for any operator  $A$ , so that  $(A^{\flat})^{\flat} = A$ . The  $\flat$ -operation is needed because it, and not the hermitian conjugation, now relates the two parts of the field,

$$[\phi^{(+)}]^{\flat} = \phi^{(-)} \neq [\phi^{(+)}]^{\dagger}. \quad (16)$$

Only at  $T = 0$ , where  $f_k = g_k = 0$ , does  $\flat = \dagger$  from (15), and there we recover the usual relationship

$$[\phi^{(+)}]^{\flat} = [\phi^{(+)}]^{\dagger} = \phi^{(-)}. \quad (17)$$

If we also enforce that the  $X_k$  operators satisfy equal time canonical commutation relations

$$[X_k(t), X_p^{\flat}(t)] = \delta_{k,p} \quad (18)$$

$$[X_k(t), X_p(t)] = 0 = [X_k^{\flat}(t), X_p^{\flat}(t)] \quad (19)$$

<sup>†</sup> Throughout this paper we use musical symbols to denote non-hermitian operations. This follows Henning [16] who uses the ‘sharp’ symbol,  $\sharp$ , to cope with the non-hermitian nature of  $\alpha \neq \frac{1}{2}$  TFD representations.

we find that the last two equations may only be satisfied if we choose  $s_k = s_{-k}$ . This determines the normalization factor,  $d_k$ , to be

$$d_k = d_{-k} = (1 + 2n_k)^{1/2}. \tag{20}$$

Thus the normalization factor which appears in the fields (12) is the square root of

$$\frac{d_k^2}{2\omega_k V} = \frac{1}{2\omega_k V} \coth\left(\frac{\beta\omega_k}{2}\right) = \frac{1}{V} \int dk_0 [\theta(k_0) + n_{|k_0|}] \delta(k_0^2 - \omega_k^2) \tag{21}$$

which is the familiar phase space density factor of finite temperature field theory.

With a view to the interpretation of the  $X_k$  operators, observe that there are some bilinear normal-ordered combinations of the  $X_k$  which are *not* zero when thermal averaged; for example

$$\langle\langle X_k^\dagger X_{-k}^\dagger \rangle\rangle = -\langle\langle X_k X_{-k} \rangle\rangle = \sqrt{n_k(1+n_k)}. \tag{22}$$

This represents a non-trivial problem in any attempt to interpret the  $X_k$  operators in terms of some type of new thermal excitation. However, such analysis is better posed in the language of bilinear transformations [17, 19] including Bogoliubov transformations. In any case the split (12) is reminiscent of structures seen with Bogoliubov transformations. We now rewrite the  $X_k$  operators in this language.

#### 4. The canonical transformation structure

From the definition of  $X_k$  and  $X_k^\dagger$ , we see that they mix  $a_k$  and  $a_{-k}^\dagger$  operators carrying opposite momentum. Such a mixing often appears in many other contexts, e.g. cosmological perturbations [20] and BCS theory [21]. The important point is that for any given  $k \neq 0$ ,  $a_k, a_k^\dagger$  commute with  $a_{-k}, a_{-k}^\dagger$ ; more generally

$$[a_k, a_p^\dagger] = \delta_{k,p}, \quad [a_k, a_p] = [a_k^\dagger, a_p^\dagger] = 0. \tag{23}$$

That is, for a given  $k \neq 0$ , we think of the sets  $\{a_k, a_k^\dagger\}$  and  $\{a_{-k}, a_{-k}^\dagger\}$  as being independent. Canonical transformations mixing these two sets [20, 21] arise naturally if we consider the Fourier amplitudes of the real field  $\phi$  and its conjugate  $\pi(x) = \dot{\phi}(x)$ ;

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_k q_k(t) e^{ik \cdot x} \tag{24}$$

$$\pi(x) = \frac{1}{\sqrt{V}} \sum_k p_k(t) e^{-ik \cdot x}. \tag{25}$$

Here

$$q_k(t) = \frac{1}{(2\omega_k)^{1/2}} (a_k(t) + a_{-k}^\dagger(t)) \quad p_k(t) = i \left(\frac{\omega_k}{2}\right)^{1/2} (a_k^\dagger(t) - a_{-k}(t)) \tag{26}$$

$$q_k^\dagger(t) = q_{-k}(t) \quad p_k^\dagger(t) = p_{-k}(t) \tag{27}$$

and

$$[q_k(t), p_{k'}(t)] = i\delta_{k,k'}. \tag{28}$$

Observe first that by definition  $p_k$  and  $q_k$  mix creation and annihilation operators with opposite momenta. Secondly, consider a *rescaling* operation of the  $q$  and  $p$  operators:  $q_k \rightarrow e^{-\theta_k} q_k$ ,  $p_k \rightarrow e^{\theta_k} p_k$ . This preserves the commutation relation (28) for *any* function  $\theta_k$ . However, in order to preserve the hermitian relationship (27) between  $q$  and  $p$  for positive and negative

$\mathbf{k}$ , one must choose  $\theta_{\mathbf{k}} = \theta_{-\mathbf{k}}$ . Then the rescaling of  $q$  and  $p$  generates a Bogoliubov transformation amongst the creation and annihilation operators;

$$a_{\mathbf{k}} \rightarrow b_{\mathbf{k}}(\theta) = e^{-iG_B(\theta)} a_{\mathbf{k}} e^{iG_B(\theta)} = a_{\mathbf{k}} \cosh \theta_{\mathbf{k}} - a_{-\mathbf{k}}^{\dagger} \sinh \theta_{\mathbf{k}} \quad (29)$$

$$a_{\mathbf{k}}^{\dagger} \rightarrow b_{\mathbf{k}}^{\dagger}(\theta) = [b_{\mathbf{k}}(\theta)]^{\dagger} \quad (30)$$

$$a_{-\mathbf{k}} \rightarrow b_{-\mathbf{k}}(\theta) = e^{-iG_B(\theta)} a_{-\mathbf{k}} e^{iG_B(\theta)} = a_{-\mathbf{k}} \cosh \theta_{\mathbf{k}} - a_{\mathbf{k}}^{\dagger} \sinh \theta_{\mathbf{k}} \quad (31)$$

$$a_{-\mathbf{k}}^{\dagger} \rightarrow b_{-\mathbf{k}}^{\dagger}(\theta) = [b_{-\mathbf{k}}(\theta)]^{\dagger} \quad (32)$$

where

$$G_B(\theta) = \frac{i}{2} \sum_{\mathbf{k}} \theta_{\mathbf{k}} [a_{\mathbf{k}}^{\dagger} a_{-\mathbf{k}}^{\dagger} - a_{-\mathbf{k}} a_{\mathbf{k}}] \quad (33)$$

and  $[b_{\mathbf{k}}(\theta), b_{\mathbf{p}}^{\dagger}(\theta)] = \delta_{\mathbf{k},\mathbf{p}}$  with all other commutators zero<sup>†</sup>. This transformation defines a new vacuum for the  $b$  operators

$$|0(\theta)\rangle = e^{-iG_B(\theta)} |0\rangle \quad b_{\mathbf{k}}(\theta) |0(\theta)\rangle = 0 \quad (34)$$

which is orthogonal to  $\langle 0|$  in the infinite volume limit. This leads to relations such as

$$\sinh^2(\theta_{\mathbf{k}}) = \langle\langle 0(\theta) | a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} | 0(\theta)\rangle\rangle = \langle\langle 0 | b_{\mathbf{k}}^{\dagger}(-\theta) b_{\mathbf{k}}(-\theta) | 0\rangle\rangle \quad (35)$$

where we have used the hermitian property of this transformation to write down the two equivalent forms. In the context of thermal field theories, the  $\alpha = \frac{1}{2}$  formulation TFD [3, 16, 17] uses a similar Bogoliubov construction. There  $\theta_{\mathbf{k}}$  is related to the particle number distribution functions through (35).

It is interesting to note, however, that rescaling  $q$  and  $p$  with an odd function of  $\theta_{\mathbf{k}}$  leads to a straight rescaling of the annihilation and creation operators,

$$a_{\mathbf{k}} \rightarrow a'_{\mathbf{k}}(\theta) = e^{\sum_{\mathbf{k}} \theta_{\mathbf{k}} N_{\mathbf{k}}} a_{\mathbf{k}} e^{-\sum_{\mathbf{k}} \theta_{\mathbf{k}} N_{\mathbf{k}}} = e^{-\theta_{\mathbf{k}}} a_{\mathbf{k}} \quad (36)$$

$$a_{\mathbf{k}}^{\dagger} \rightarrow a'^{\natural}_{\mathbf{k}}(\theta) = e^{\sum_{\mathbf{k}} \theta_{\mathbf{k}} N_{\mathbf{k}}} a_{\mathbf{k}}^{\dagger} e^{-\sum_{\mathbf{k}} \theta_{\mathbf{k}} N_{\mathbf{k}}} = e^{\theta_{\mathbf{k}}} a_{\mathbf{k}}^{\dagger} \quad (37)$$

where the sum is over all momenta, and  $N_{\mathbf{k}} = a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}$ . Note that unlike the case of the Bogoliubov transformation (33), this rescaling transformation does *not* mix operators with opposite momentum. Thus in fact (36) and (37) hold for all momenta and for any function  $\theta_{\mathbf{k}}$ . However, only for odd functions does it correspond to a rescaling of  $q$  and  $p$ . The important point to note is that unlike the Bogoliubov transformation, this transformation is *not* hermitian, and as a result it is not usually discussed. It does though lead to a canonical set of operators;  $[a'_{\mathbf{k}}(\theta), a'^{\natural}_{\mathbf{p}}(\theta)] = \delta_{\mathbf{k},\mathbf{p}}$  with all other commutators zero. Here the ‘natural conjugation’,  $\natural$ , consists of hermitian conjugation and the replacement  $\theta \leftrightarrow -\theta$  (37). Amusingly, if we put  $\theta_{\mathbf{k}} = \omega_{\mathbf{k}} t_4$ , this rescaling is a Euclidean time translation by  $t_4$ , and such rescalings play a key role in all path-ordered approaches to thermal field theory. They also appear in TFD for  $\alpha \neq \frac{1}{2}$  formulations [16]. Thus rescaling transformations are in fact extremely common.

Having turned away from hermitian transformations, we can return to the Bogoliubov transformation (29)–(32) and note that there is another closely related canonical pair related by the  $\natural$  operation:

$$a_{\mathbf{k}} \rightarrow c_{\mathbf{k}}(\theta) = \frac{1}{m_{\mathbf{k}}} b_{\mathbf{k}}(\theta) = \frac{1}{m_{\mathbf{k}}} e^{-iG_B(\theta)} a_{\mathbf{k}} e^{iG_B(\theta)} = \frac{1}{m_{\mathbf{k}}} [a_{\mathbf{k}} \cosh \theta_{\mathbf{k}} - a_{-\mathbf{k}}^{\dagger} \sinh \theta_{\mathbf{k}}] \quad (38)$$

$$a_{-\mathbf{k}}^{\dagger} \rightarrow c_{-\mathbf{k}}^{\natural}(\theta) = \frac{1}{m_{\mathbf{k}}} [b_{-\mathbf{k}}(-\theta)]^{\dagger} = \frac{1}{m_{\mathbf{k}}} e^{iG_B(\theta)} a_{-\mathbf{k}}^{\dagger} e^{-iG_B(\theta)}$$

<sup>†</sup> The factor of  $\frac{1}{2}$  is needed because  $\theta_{\mathbf{k}}$  and  $\theta_{-\mathbf{k}}$  are regarded as the *same* independent variable. The point  $\mathbf{k} = 0$  is omitted from the summation.

$$= \frac{1}{m_k} [a_k \sinh \theta_k + a_{-k}^\dagger \cosh \theta_k] \tag{39}$$

$$\begin{aligned} a_{-k} \rightarrow c_{-k}(\theta) &= \frac{1}{m_k} b_{-k}(\theta) = \frac{1}{m_k} e^{-iG_B(\theta)} a_{-k} e^{iG_B(\theta)} \\ &= \frac{1}{m_k} [a_{-k} \cosh \theta_k - a_k^\dagger \sinh \theta_k] \end{aligned} \tag{40}$$

$$a_k^\dagger \rightarrow c_k^\dagger = \frac{1}{m_k} (b_k(-\theta))^\dagger = \frac{1}{m_k} e^{iG_B(\theta)} a_k^\dagger e^{-iG_B(\theta)} = \frac{1}{m_k} [a_{-k} \sinh \theta_k + a_k^\dagger \cosh \theta_k] \tag{41}$$

with

$$m_k = m_{-k} = [\sinh^2 \theta_k + \cosh^2 \theta_k]^{1/2} \tag{42}$$

where we have used  $G_B(-\theta) = -G_B(\theta)$ . Observe that the  $\natural$  conjugation is defined as above, and that it is responsible for the change of sign in front of  $\sinh \theta$  terms in moving from (38) to (39), or from (40) to (41). It is also responsible for the origin of the normalization,  $m_k$ . This transformation preserves all the commutation relations

$$[c_k, c_p^\natural] = \delta_{k,p} \quad [c_k, c_p] = [c_k^\natural, c_p^\natural] = 0. \tag{43}$$

Now we return to the case of the  $X_k$  operators (14). The analysis is made much simpler by introducing a specific temperature dependent angle  $\sigma_k = \sigma_{-k}$  to parameterize the  $T > 0$  split defined by (10). Note that  $f_k$  and  $g_k$  involve the Bose–Einstein distributions through factors like  $n_k, 1 + n_k$  and their square roots. In many situations in thermal field theory the special properties of these distributions are crucial and may be encoded by the use of hyperbolic functions. Thus we are led to write

$$n_k = \sinh^2(\sigma_k) = \frac{1}{e^{\beta\omega_k} - 1} \tag{44}$$

though this does not fix the sign of  $\sigma_k$ . Again, TFD is especially inspirational as it uses the same parameterization for the Bose–Einstein distribution  $n_k$  [3–5, 16, 17] though it applies it in a different way.

In terms of  $\sigma_k$ , the functions  $f_k$  and  $g_k$  of (10) can be rewritten as

$$f_k = e^{-\sigma_k} \sinh(\sigma_k) \quad g_k = -e^{\sigma_k} \sinh(\sigma_k) \tag{45}$$

$$(1 - f_k) = e^{-\sigma_k} \cosh(\sigma_k) \quad (1 - g_k) = e^{\sigma_k} \cosh(\sigma_k). \tag{46}$$

This shows that changing the sign of  $\sigma_k$  is equivalent to swapping the  $f_k$  and  $g_k$  functions. Thus  $\flat$ -conjugation is hermitian conjugation plus the exchange  $\sigma_k \leftrightarrow -\sigma_k$ ; that is, for some operator  $A[\sigma]$  we have

$$(A[\sigma_k])^\flat = (A[-\sigma_k])^\dagger. \tag{47}$$

Thus we see that the  $\flat$  and  $\natural$  operation are identical if we set  $\theta = \sigma$  in the definition of  $\natural$ .

We may now re-write the  $X_k$  operators of (14) in terms of  $\sigma_k$  (dropping the  $t$ -dependence for notational simplicity);

$$X_k = \frac{1}{d_k} (\cosh(\sigma_k) e^{-\sigma_k} a_k - \sinh(\sigma_k) e^{\sigma_k} a_{-k}^\dagger) \tag{48}$$

$$X_k^\flat = \frac{1}{d_k} (\sinh(\sigma_k) e^{-\sigma_k} a_{-k} + \cosh(\sigma_k) e^{\sigma_k} a_k^\dagger) \tag{49}$$

where  $\sigma_k = +\sigma_{-k}, s_k = +s_{-k}$ . From these equations it is straightforward to see that we can now view the  $X_k$  operator as being generated by the combination of a scaling transformation, which generates the parts  $e^{-\sigma_k} a_k$  and  $e^{\sigma_k} a_{-k}^\dagger$  of (48) according to (36) and (37), and the new



transformation of (38)–(41). Thus, depending on the order in which we carry out these two operations, we may write

$$X_k = \frac{1}{d_k} (e^{-iG'_B(\sigma)} e^{\sum_k \sigma_k N_k} a_k e^{-\sum_k \sigma_k N_k} e^{iG'_B(\sigma)}) \quad (50)$$

$$= \frac{1}{d_k} (e^{\sum_k \sigma_k N_k} e^{-iG_B(\sigma)} a_k e^{iG_B(\sigma)} e^{-\sum_k \sigma_k N_k}) \quad (51)$$

and

$$X_k^{\flat} = \frac{1}{d_k} (e^{iG'_B(\sigma)} e^{\sum_k \sigma_k N_k} a_k^{\dagger} e^{-\sum_k \sigma_k N_k} e^{-iG'_B(\sigma)}) \quad (52)$$

$$= \frac{1}{d_k} (e^{\sum_k \sigma_k N_k} e^{iG_B(\sigma)} a_k^{\dagger} e^{-iG_B(\sigma)} e^{-\sum_k \sigma_k N_k}) \quad (53)$$

with

$$d_k = d_{-k} = (\cosh(2\sigma_k))^{1/2} = (1 + 2n_k)^{1/2}. \quad (54)$$

Here

$$G'_B(\sigma) = \frac{i}{2} \sum_k \sigma_k [a_k^{\flat} a_{-k}^{\flat} - a'_{-k} a'_k] = -(G_B^{\flat}(\sigma))^{\flat} \quad (55)$$

where  $a'_k$  and  $a_k^{\flat}$  are implicitly functions of  $\sigma$  and are defined in (36) and (37) (with  $\theta \rightarrow \sigma$ ,  $\flat \rightarrow \flat$ ). Thus, finally, we see that the new  $X_k$  are related to the original  $a_k$  by a combination of a Bogoliubov-like transformation (38)–(41) and a scaling transformation (36), (37).

We can also construct the vacuum,  $|0_X(t)\rangle$ , for the  $X_k$  operators

$$X_k(t)|0_X(t)\rangle = \langle 0_X(t)|X_k^{\flat}(t) = 0. \quad (56)$$

From (50) we see that the ket vacuum must be

$$|0_X(t)\rangle = N e^{-iG'_B(\sigma)} |0\rangle \quad (57)$$

$$= N \prod_k \frac{1}{\cosh(\sigma_k)} \exp[\tanh(\sigma_k) a_k^{\flat}(t) a'_{-k}(t)] |0\rangle. \quad (58)$$

The  $X$  vacuum is therefore a condensate of zero-momentum pairs of  $a$ -particles. In defining the bra vacuum, care must be taken to ensure  $\flat$ -conjugation is consistently used rather than hermitian conjugation. Thus from (52) we can define

$$\langle 0_X(t)| = (|0_X(t)\rangle)^{\flat} \neq (|0_X(t)\rangle)^{\dagger} = \langle 0| e^{+iG'_B(\sigma)} N \quad (59)$$

$$= \langle 0| e^{-iG'_B(\sigma)} N$$

$$= \langle 0| N \prod_k \frac{1}{\cosh(\sigma_k)} \exp[-\tanh(\sigma_k) a'_k(t) a'_{-k}(t)]. \quad (60)$$

The normalization factor  $N$  is not trivial and we find

$$\langle 0_X(t)|0_X(t)\rangle = 1 \Rightarrow N = \prod_k d_k^2. \quad (61)$$

In the infinite volume limit we have

$$\langle 0_X(t)|0\rangle = \langle 0|0_X(t)\rangle \rightarrow 0 \quad \forall t \quad V \rightarrow \infty. \quad (62)$$

We now have the full structure of the new operators  $X_k$  and their associated Fock space. At this stage in the usual examples of Bogoliubov transformations (symmetry breaking, TFD, etc) we would show that the physical vacuum was the transformed vacuum and not that associated

with the physical operators  $a$ . When we attempt to do this here, we find that for a general operator  $A[a, a^\dagger]$  we have

$$\langle 0_X(t) | A[a, a^\dagger] | 0_X(t) \rangle \neq \langle \langle A \rangle \rangle. \tag{63}$$

For example,

$$\frac{1}{\langle 0_X(t) | 0_X(t) \rangle} \langle 0_X(t) | a_k^\dagger a_k | 0_X(t) \rangle = \frac{-1}{\exp(\beta\omega_k) + 1} \neq n_k \tag{64}$$

$$\frac{1}{\langle 0_X(t) | 0_X(t) \rangle} \langle 0_X(t) | a_k a_k^\dagger | 0_X(t) \rangle = \frac{1}{\exp(-\beta\omega_k) + 1} \neq 1 + n_k. \tag{65}$$

Hence the vector  $|0_X(t)\rangle$  (and  $\langle 0_X(t)|$ ) is not a thermal vacuum.

### 5. Conclusion and outlook

In this paper we have discovered a new set of canonical operators for real scalar fields in thermal equilibrium—the  $X_k$  of (48) and (49)—for all path-ordered approaches to thermal field theory such as Matsubara’s imaginary time-formalism. These follow from the redefinition of the normal-ordered product found necessary if the canonical approach path-ordered thermal field theory is to proceed in the usual way [1]. We have then shown that the  $X_k$  are produced from the original creation and annihilation operators by a pair of transformations, one Bogoliubov-like and one rescaling (50) and (53). Finding these transformations is not trivial since the conjugate pair  $X_k$  and  $X_k^b$  are not related by hermitian conjugation but by our flat conjugation, (47), so we have had to look beyond the standard transformations of the literature.

Through the use of Bogoliubov transformations in QFT, one is usually able to talk in terms of quasi-particles, or in the case of TFD one is able to replace the thermal trace with a thermal vacuum. Unfortunately, both the lack of a hermitian structure and the inability to duplicate the thermal trace results using the vacuum of the  $X_k$  operators means that it is difficult to give a meaning in terms of physical excitations to this new set of canonical operators  $X_k$ . This is not surprising in view of the mentioned difficulties of defining the particle concept at finite temperature. What is relevant, and in some sense it is our main result, is that nevertheless, the formalism may support a canonical transformation structure. Hence, as we have shown, well-defined vacuum state (and associated Hilbert space) exists. However, such a vacuum is not the thermal vacuum. At same time, the split in terms of the  $X_k$ , equations (8), (9) and (12), guarantees that the thermal averages of all  $n$ -point normal-ordered products of fields vanish, which in turn makes the path-ordered approach to thermal field theory proceed in the usual way [1].

Finally, there are some other results which are worth noting. There are in fact several alternative sets of canonical operators based on the thermal normal-ordered product. One could just mix  $a_k$  and  $a_k^\dagger$  but keep the same coefficients rather than mix  $a_k$  and  $a_{-k}^\dagger$  as we considered in (14). This, however, would make the  $X_k$  contain a specific position dependence,  $X(x)$ . Alternatively, one can perform a further rescaling of the  $X_k$  given here,  $X_k \rightarrow \exp\{\theta_k\} X_k$ ,  $X_k^b \rightarrow \exp\{-\theta_k\} X_k^b$  for any  $\theta_k$  and keep the commutation relations.

Most interestingly one can exploit the freedom in the sign of  $\sigma_k$  and work with an odd function rather than the even function we have chosen here (equivalent to  $s_k = -s_{-k}$  in (10)). Still using the idea that annihilation and creation operators of opposite momenta form mutually commuting sets, we mix just the annihilation operators (or just creation operators) of opposite momentum to form new canonical operators,  $W$ . This technique works for all types of bosonic field. For example, in the case of non-relativistic fields we can define four new canonical operators  $W_1, W_2, W_2^b$  and  $W_1^b$  based on the split given in [1]. There is some flexibility in the

notation one can use. If we stick with the flat conjugation definition (47) and demand that normal ordering of (7) and [1] puts the annihilation operators  $W_1, W_2$  to the right and creation operators  $W_1^\dagger, W_2^\dagger$  to the left, then we find that

$$\psi^{(+)}(x) = \sum_k \frac{1}{\sqrt{2\omega_k V}} [(1 - f_k) a_k e^{-i\omega t} e^{ik \cdot x}] = \sum_{k>0} \frac{\cosh(\sigma_k)}{(\omega_k V)^{\frac{1}{2}}} W_1(\mathbf{k}, x) e^{-i\omega t} \quad (66)$$

$$\psi^{(-)}(x) = \sum_k \frac{1}{\sqrt{2\omega_k V}} [f_k a_k e^{-i\omega_k t} e^{ik \cdot x}] = \sum_{k>0} \frac{\sinh(\sigma_k)}{(\omega_k V)^{\frac{1}{2}}} W_2^\dagger(\mathbf{k}, x) e^{-i\omega_k t} \quad (67)$$

$$\psi^{\dagger(+)}(x) = \sum_k \frac{1}{\sqrt{2\omega_k V}} [g_k a_k^\dagger e^{-i\omega_k t} e^{ik \cdot x}] = \sum_{k>0} \frac{\sinh(\sigma_k)}{(\omega_k V)^{\frac{1}{2}}} [-W_2(\mathbf{k}, x)] e^{-i\omega_k t} \quad (68)$$

$$\psi^{\dagger(-)}(x) = \sum_k \frac{1}{\sqrt{2\omega_k V}} [(1 - g_k) a_k^\dagger e^{-i\omega_k t} e^{ik \cdot x}] = \sum_{k>0} \frac{\cosh(\sigma_k)}{(\omega_k V)^{\frac{1}{2}}} W_1^\dagger(\mathbf{k}, x) e^{-i\omega_k t} \quad (69)$$

where the sum over  $k > 0$  indicates that we are summing over half of  $k$  space, including only one of each  $(\mathbf{k}, -\mathbf{k})$  pair. The commutation relations satisfied by the  $W$  are then seen to be

$$[W_i(\mathbf{k}, x), W_j^\dagger(\mathbf{p}, x)] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \delta_{\mathbf{k}, \mathbf{p}} \quad (70)$$

with other commutators zero. We are currently investigating the structure defined by these operators and its relation to that defined by the  $X_k$ .

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## References

- [1] Evans T S and Steer D A 1996 *Nucl. Phys. B* **474** 481
- [2] Matsubara T 1955 *Prog. Theor. Phys.* **14** 4
- [3] Rivers R J 1987 *Path Integral Methods in Quantum Field Theory* (Cambridge: Cambridge University Press)
- [4] Landsman N P and van Weert Ch G 1987 *Phys. Rep.* **145** 141
- [5] van Weert Ch G 1994 *Proc. of the Banff/CAP Workshop on Thermal Field Theories* ed F C Khanna et al (Singapore: World Scientific) p 1
- [6] Le Bellac M 1996 *Thermal Field Theory* (Cambridge: Cambridge University Press)
- [7] Strocchi F 1985 *Elements of Quantum Mechanics of Infinite Systems* (Singapore: World Scientific)
- [8] Blasone M, Henning P A and Vitiello G 1998 *Preprint* hep-th/9803157  
Alfinito E, Blasone M, Iorio A and Vitiello G 1996 *Acta Phys. Pol. B* **27** 1493  
Alfinito E, Blasone M, Iorio A and Vitiello G 1995 *Phys. Lett. B* **362** 91  
Blasone M and Vitiello G 1995 *Ann. Phys., NY* **244** 283
- [9] Martellini M, Sodano P and Vitiello G 1978 *Nuovo Cimento A* **48** 341
- [10] Birrell N D and Davis P C W 1988 *Quantum Field in Curved Space Time* (Cambridge: Cambridge University Press)
- [11] Alfinito E, Manka R and Vitiello G 1997 *Preprint* hep-th/9705134
- [12] Celeghini E, Rasetti M and Vitiello G 1992 *Ann. Phys., NY* **215** 156  
Iorio A and Vitiello G 1995 *Ann. Phys., NY* **241** 496  
Srivastava Y N, Vitiello G and Widom A 1995 *Ann. Phys., NY* **238** 200
- [13] Bogoliubov N N 1958 *Sov. Phys. - JETP* **7** 41  
Valatin J G 1958 *Nuovo Cimento* **7** 843
- [14] Jackiw R 1995 Two lectures on two-dimensional gravity *Preprint* gr-qc/9511048  
Cangemi D, Jackiw R and Zwiebach B 1996 *Ann. Phys., NY* **245** 408

- [15] Landsman N P 1988 *Ann. Phys., NY* **186** 141
- [16] Henning P A 1995 *Phys. Rep.* **253** 235
- [17] Takahashi Y and Umezawa H 1975 *Collective Phenom.* **2** 55  
Umezawa H, Matsumoto H and Tachiki M 1982 *Thermo Field Dynamics and Condensed States* (Amsterdam: North Holland)
- [18] Schmutz M 1978 *Z. Phys. B* **30** 97  
Lawrie I D 1994 *J. Phys. A: Math. Gen.* **27** 1435
- [19] Barnett S and Knight P 1985 *J. Opt. Soc. Am. B* **21** 467  
Schumaker B L 1986 *Phys. Rep.* **135** 317
- [20] Prokopec T 1993 *Class. Quantum Grav.* **10** 2295
- [21] Fetter A and Walecka J 1971 *Quantum Theory of Many-Particle Systems* (New York: McGraw-Hill)